Brightness Function Program Information

1. Contributors and history.

The underlying algorithm was published by Richard J. Gardner and Peyman Milanfar [3] in 2001. The same year, it was implemented with the assistance of Western Washington University (WWU) undergraduate student Chris Street. From 2003 to 2006, improvements were made by WWU undergraduate students Chris Eastman, Thomas Riehle, and Greg Richardson. The GUI was designed by WWU undergraduate student Dale Jennings in 2010. The original web page GUI was designed by WWU undergraduate student Dale Jennings in 2010 and subsequently modified by WWU undergraduate students Roberto Vergaray, Ian Fisk, and Elliott Skomski.

The convergence property of the algorithm was proved for exact data by Gardner and Milanfar [4] and for noisy data by Gardner, Kiderlen, and Milanfar [5].

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2. How the program works.

A fairly comprehensive description of the algorithm can be found in [2, Section 4.4 and Note 4.12], so we only provide a brief outline here.

The underlying theoretical result is Aleksandrov's projection theorem [2, Theorem 3.3.6], which says that an origin-symmetric convex body in $\mathbb{R}^n$ is uniquely determined by its brightness function, i.e., by the areas of its orthogonal projections ("shadows") on hyperplanes. The algorithm works in two phases: In Phase I, the surface area measure of an origin-symmetric convex polytope that approximates the target shape is obtained from finitely many noisy measurements of the brightness function, and in Phase II, this polytope is reconstructed from the surface area measure. (Phase I was considered earlier, though in a quite different context, by Kiderlen [6], [7].)

Phase I works by using Cauchy's projection formula [2, (A.45), p. 408] to compute the areas of shadows of a convex polytope described in terms of the areas of and outer unit normal vectors to its facets. A least-squares approach is used to find the origin-symmetric convex polytope that best fits the data. (An alternative linear programming method, appropriate when noise is absent or very low, is also available.) A key point is that a suitable set of outer unit normal vectors to the facets can be prescribed in advance and computed from the measurement directions; this is an application of a theoretical result due to Campi, Colesanti, and Gronchi [1]. In view of this, the only variables are the areas of the facets, which appear in a linear fashion. The optimization problem is solved by the function `lsqnonneg` from Matlab's Optimization Toolbox.

Phase II is essentially Algorithm MinkData, the algorithm for reconstructing a convex polytope from its surface area measure. See the Geometric Tomography web page for more information.

For reconstructions via the Geometric Tomography web page, the user can choose the number of measurement directions, up to 50. For each such choice the directions are taken from a library of the best known chordal packings of lines available on the web page of Neil Sloane at `http://neilsloane.com/grass/dim3/`. This ensures that the measurement directions are as "spread out" as possible over the unit sphere.

Many other features have been implemented beyond those offered in the present GUI. Some of these may be made available in future versions.

References

[1] S. Campi. A. Colesanti, and P. Gronchi, Convex bodies with extremal volumes having prescribed brightness in finitely many directions, *Geom. Dedicata* **57** (1995), 121–33.

[2] R. J. Gardner, *Geometric Tomography*, second edition, Cambridge University Press, New York, 2006.

[3] R. J. Gardner and Peyman Milanfar, Shape reconstruction from brightness functions, in: *Proceedings of SPIE Conference on Advanced Signal Processing Algorithms, Architectures, and Implementations X*, San Diego, CA, 2001, Proceedings of SPIE **4474**, pp. 234–245.

[4] R. J. Gardner and P. Milanfar, Reconstruction of convex bodies from brightness functions, *Discrete Comput. Geom.* **29** (2003), 279–303.

[5] R. J. Gardner, M. Kiderlen, and P. Milanfar, Convergence of algorithms for reconstructing convex bodies and directional measures, *Ann. Statist.* **34** (2006), 1331–1374.

[6] M. Kiderlen, Non-parametric estimation of the directional distribution of stationary line and fibre processes, *Adv. in Appl. Probab.* **33** (2001), 6–24.

[7] M. Kiderlen and A. Pfrang, Algorithms to estimate the rose of directions of a spatial fiber system, *J. Microsc.* **219** (2005), 50–60.