Covariogram Program Information

1. Contributors and history.

The underlying algorithm was conceived by Gabriele Bianchi, Richard J. Gardner, and Markus Kiderlen in 2008 and published, along with a proof of convergence, in [1]. In 2008–9, it was implemented with the assistance of Western Washington University (WWU) undergraduate student Michael Sterling-Goens. Improvements were made in 2009–10 by WWU undergraduate student Nick Henscheid. The original web page GUI was designed by WWU undergraduate student Dale Jennings in 2010 and subsequently modified by WWU undergraduate students Roberto Vergaray, Ian Fisk, and Elliott Skomski.

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2. How the program works.

We begin with a description, adapted from [1], that applies to reconstruction from covariograms in $\mathbb{R}^n$ for any $n \geq 2$. After that, we discuss how this specializes to the planar case $n = 2$ implemented here.

A brief background on covariograms can be found under "Algorithms" in the text part of the Geometric Tomography web page. The algorithm takes as input a finite number of noisy measurements of the covariogram g_K of an unknown convex body K. It is assumed that K is uniquely determined (up to translation and reflection in the origin) by its covariogram, has its centroid at the origin, and is contained in a known bounded region C of $\mathbb{R}^n$. In general, for each suitable $k \in \mathbb{N}$, a convex polytope P_k is constructed that approximates K or its reflection $-K$. There are two phases: an initial phase that produces suitable outer unit normals to the facets of P_k, and a main phase that goes on to actually construct P_k.

In the first phase, the covariogram of K is measured, *multiple times*, at the origin and at vectors $(1/k)u_i$, $i = 1, \dots, k$, where the u_i's are mutually nonparallel unit vectors that span $\mathbb{R}^n$. From these measurements, an algorithm called Algorithm NoisyCovBlaschke constructs an origin-symmetric convex polytope Q_k that approximates ∇K, the so-called Blaschke body of K. The definition of the Blaschke body can be found in [2, p. 116], but the crucial property of ∇K is that when K is a convex polytope, each of its facets is parallel to some facet of ∇K. It follows that the outer unit normals to the facets of P_k can be taken to be among those of Q_k. Algorithm NoisyCovBlaschke utilizes the known fact that $-\partial g_K(tu)/\partial t$, evaluated at $t = 0$, equals the brightness function value $b_K(u)$, that is, the $(n-1)$-dimensional volume of the orthogonal projection of K in the direction u. Thus the measurements provide an approximation to $b_K(u_i)$ for each i, where larger k's give better approximations. This connection allows most of the work to be done by a very efficient algorithm, Algorithm NoisyBrightLSQ, designed earlier by Gardner and Milanfar for reconstructing an origin-symmetric convex body from finitely many noisy measurements of its brightness function. Algorithm NoisyBrightLSQ is described under "Algorithms" on the Geometric Tomography web page and is also implemented here when $n = 3$ (the "Brightness Function" button under "Algorithms").

The output Q_k of the first phase forms part of the input to the main phase, called Algorithm NoisyCovLSQ. The covariogram of K is now measured *again*; assuming, for example, that C is the cube $[-1/2, 1/2]^n$, it is measured once at each point in a cubic array in $2C = [-1, 1]^n$ of side length $1/k$. Using these measurements, Algorithm NoisyCovLSQ finds a convex polytope P_k, each of whose facets is parallel to some facet of Q_k, whose covariogram fits best the measurements in the least squares sense.

All this works, in principle, in any dimension, but presently the algorithm is only implemented in the plane, when $n = 2$. It is known that any planar convex body is determined by its covariogram, so the uniqueness criterion is automatically satisfied. A considerably simpler version of the Brightness Function program can be employed; in fact, in the plane, reconstructing an origin-symmetric convex body from its brightness function is equivalent to reconstructing it from its support function.

Other features have been implemented beyond those offered in the present GUI. For example, suitable values of k (different for the two phases mentioned above) have been chosen and cannot be changed by the user. Some of the missing features may be made available in future versions.

References

[1] Gabriele Bianchi, R. J. Gardner, and Markus Kiderlen, Phase retrieval for characteristic functions of convex bodies and reconstruction from covariograms, *J. Amer. Math. Soc.* **24** (2011), 293–343.

[2] R. J. Gardner, *Geometric Tomography*, second edition, Cambridge University Press, New York, 2006.