Support Function Program Information

1. **Contributors and history.**

The underlying algorithm was conceived by Richard J. Gardner and Markus Kiderlen in 2004 and published in [2]. It was implemented in 2006–7 with the assistance of Western Washington University (WWU) undergraduates Greg Richardson, LeRoy Miller, and Michael Taron. The original web page GUI was designed by WWU undergraduate student Dale Jennings in 2010 and subsequently modified by WWU undergraduate students Roberto Vergaray, Ian Fisk, and Elliott Skomski.

Convergence of the algorithm was proved by Gardner, Kiderlen, and Milanfar [3].

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2. **How the program works.**

If the target shape is a convex body K in $\mathbb{R}^n$, noisy measurements $y_1, \ldots, y_k$ of the support function h_K of K are taken in directions $u_1, \ldots, u_k$, respectively. The main action is then to solve the following constrained linear least squares problem:

$$\min_{x_1 \in \mathbb{R}^n, \ldots, x_k \in \mathbb{R}^n} \quad \sum_{i=1}^{k} (y_i - x_i^T u_i)^2, \tag{1}$$

$$\text{subject to} \quad x_j^T u_i \le x_i^T u_i \text{ for } 1 \le i \ne j \le k. \tag{2}$$

If $\hat{x}_1, \ldots, \hat{x}_k$ is a solution of (1)–(2), then the output of the program is the convex hull $\hat{Q}_k$ of $\{\hat{x}_1, \ldots, \hat{x}_k\}$.

To understand how this works, note that for each direction u_i, there is at least one point x_i in K contained in the supporting (tangent) hyperplane

$$\{x \in \mathbb{R}^n : x^T u_i = h_K(u_i)\}.$$

By the definitions of the support function and supporting hyperplane, when $1 \le j \le k$, we have

$$x_j^T u_i \le h_K(u_i) = x_i^T u_i,$$

so the constraint (2) is satisfied. If the measurements are exact, then $\hat{Q}_k$ is a convex polytope with the same support function values as K in the measurement directions $u_1, \ldots, u_k$; see Figure 1 for a two-dimensional illustration.

Implementation is done with Matlab's Optimization Toolbox to solve the least squares problem, and Matlab's convex hull function and graphics to produce the picture for three-dimensional reconstruction.

A linear program version of the algorithm is also provided (see [2] for details), and this is much faster and better, or at least comparable, in performance at low levels of noise and reasonably small numbers of measurements.

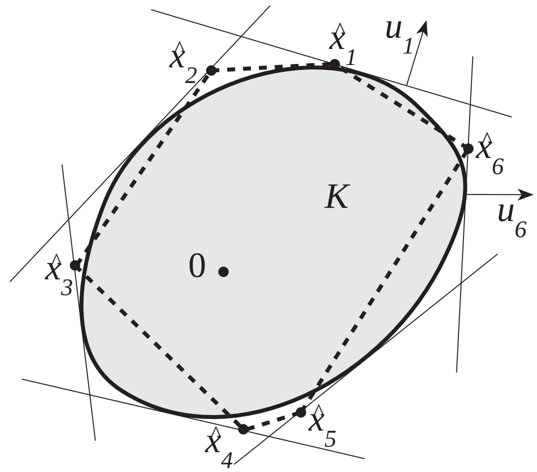

Figure 1. Possible output (dotted) of the algorithm when measurements are exact (shown in two dimensions for convenience).

It is perhaps surprising that the algorithm is the first fully effective one for the purpose of reconstruction from support functions in $\mathbb{R}^3$. In 1990, Prince and Willsky [4] proposed and implemented an algorithm, also based on a constrained linear least-squares fit, that works very well in two dimensions. However, it fails to be effective in $\mathbb{R}^3$ because its constraint is hard to make explicit. In the algorithm used here, the consistency constraint is always completely explicit in (2), and indeed requires only $k(k-1)$ linear inequality constraints. On the other hand, there are $3k$ real variables in the objective function (1) when $n = 3$, as opposed to k in the Prince-Willsky algorithm.

For background on reconstruction from support functions, including an explicit description of the Prince-Willsky algorithm, see [2] or [1, Note 3.8].

For reconstructions via the Geometric Tomography web page, the user can choose the number of measurement directions, up to 50. For each such choice the directions are taken from a library of the best known chordal packings of lines available on the web page of Neil Sloane at `http://neilsloane.com/grass/dim3/`.

Many other features have been implemented beyond those offered in the present GUI. Some of these may be made available in future versions.

References

[1] R. J. Gardner, *Geometric Tomography*, second edition, Cambridge University Press, New York, 2006.

[2] R. J. Gardner and Markus Kiderlen, A new algorithm for 3D reconstruction from support functions, *IEEE Trans. Pattern Anal. Machine Intell.* **31** (2009), 556–562.

[3] R. J. Gardner, M. Kiderlen, and P. Milanfar, Convergence of algorithms for reconstructing convex bodies and directional measures, *Ann. Statist.* **34** (2006), 1331–1374.

[4] J. L. Prince and A. S. Willsky, Estimating convex sets from noisy support line measurements, *IEEE Trans. Pattern Anal. Machine Intell.* **12** (1990), 377–389.